HOMEWORK 11

1) (20pts) Prove or give a counterexample: if V is finite-dimensional and U is a subspace of V that is invariant under every $T \in \mathcal{L}(V, V)$, then $U = \{0\}$ or $U = V$.

2) (20pts) Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be defined by $T(x_1, x_2, \ldots, x_n) = (x_1, 2x_2, \ldots, nx_n)$.

(a) Find all eigenvalues and the associated eigenvector spaces of T.

(b) Find all invariant subspaces of T.

3) (20pts) Suppose $V = U \oplus W$, where U and W are non-zero subspaces of V. Define $P \in \mathcal{L}(V, V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P.

4) (20pts) Suppose $T \in \mathcal{L}(V, V)$ and $\dim(\text{Range}(T)) = k$. Prove that T has at most $k+1$ distinct eigenvalues.

5) (20pts) Suppose $A \in \mathrm{M}_n(\mathbb{F})$ and define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ by $T(v) = Av$, where the elements of $\mathbb{F}^n$ are thought of as $n \times 1$ column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.