Homework 10

1) (30pts) Let $A \in \mathrm{M}_3(\mathbb{C})$. Consider the matrix $xId - A$ with polynomial entries.

(a) Show that $\det(xId - A)$ is a monic polynomial of degree 3.

(b) If $\det(xId - A) = (x - c_1)(x - c_2)(x - c_3)$ with $c_1, c_2, c_3 \in \mathbb{C}$ the roots of $\det(xId - A)$, prove that

$$c_1 + c_2 + c_3 = \mathrm{trace}(A) \quad \text{and} \quad c_1 c_2 c_3 = \det(A).$$

2) (20pts) Find the inverse of the following matrix using the adjoint:

$$A = \begin{pmatrix} -2 & 3 & 2 & -6 \\ 0 & 4 & 4 & -5 \\ 5 & -6 & -3 & 2 \\ -3 & 7 & 0 & 0 \end{pmatrix}.$$

3) (30pts) Prove by induction that if $k_1 \dots, k_n \in \mathbb{F}$, then we have

$$\det \begin{pmatrix} 1 + k_1 & k_2 & k_3 & \cdots & k_n \\ k_1 & 1 + k_2 & k_3 & \cdots & k_n \\ k_1 & k_2 & 1 + k_3 & \cdots & k_n \\ \vdots & \vdots & \vdots & & \vdots \\ k_1 & k_2 & k_3 & \cdots & 1 + k_n \end{pmatrix} = 1 + k_1 + \cdots + k_n.$$

4) (20pts) A matrix $A \in \mathrm{M}_n(\mathbb{R})$ is called *skew-symmetric* if $A^T = -A$.

(a) Prove that if n is odd and $A \in \mathrm{M}_n(\mathbb{R})$ is skew-symmetric, then $\det(A) = 0$.

(b) For every even n, find a skew-symmetric matrix $A \in \mathrm{M}_n(\mathbb{R})$ such that $\det(A) \neq 0$.