Student Feedback Report

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Lecturer	Paulo TIRAO

Open question score	Original Exam Grade	**Final Exam Grade**
89.00	89.00	**89.00**

Summary

Question number	Actual points	Max points
1.1	4.00	4.00
1.2	5.00	8.00
1.3	8.00	8.00
2.1	7.00	7.00
2.2	3.00	3.00
2.3	5.00	7.00
2.4	3.00	3.00
3.1	5.00	5.00
3.2	5.00	5.00
3.3	5.00	5.00
3.4	0	5.00
4.1	9.00	10.00
4.2	10.00	10.00
5.1	6.00	6.00
5.2	7.00	7.00
5.3	7.00	7.00

✓ Correct answer ✓ Partial answer × Incorrect answer ⦸ Not answered

(60)

ID 999027873

Exam 202411301040665

Guangdong Technion
Israel Institute of Technology
广东以色列理工学院

Algebra A - 104066

MIDTERM - November 30, 2024

Your ID Number: 9 9 9 0 2 7 8 7 3

First name and surname: Shi Yue / Yue Shi

- **Duration: 120 minutes**.
- Use of calculators, personal dictionaries, electronic devices, reference materials, personal notes or any other extra material is not allowed.
- Present your answers below the corresponding statement, and use extra pages if needed.
- Write complete, rigorous and correct arguments in every item.
- Presentation and writing will be evaluated as well. Write with no scratches.

PROBLEM 1 (20 POINTS). **On finite dimensional vector spaces.**

Let V be an $\mathbb{F}$-vector space.

(a) Define when V is said to be finite-dimensional.

(b) Prove that if $v_1, \ldots, v_k$ is a linearly dependent list in V, then there exists $1 \leq j \leq k$ such that:

  (a) $v_j \in \langle v_1, \ldots, v_{j-1} \rangle$ and

  (b) $\langle v_1, \ldots, \widehat{v_j}, \ldots, v_k \rangle = \langle v_1, \ldots, v_k \rangle$.

(c) Prove that every finite-dimensional vector space has a basis.

(a) V (an F-vector space) is spanned/generated by a list of finite vectors, which is spanning list. Then V is said to be finite dimensional

4
(1.1)

(b) (a) Since $V_1, \ldots, V_k$ is a linearly dependent, then for $a_1v_1 + a_2v_2 + \ldots + a_kV_k = 0$, $\exists$ one a is not 0.

Let's call the first non-zero element a. be $\boxed{a_j}$

Then $a_1v_1 + a_2v_2 + \cdots + a_{j-1}V_{j-1} = -a_jV_j$

$\Rightarrow V_j = -\frac{a_1}{a_j}V_1 - \frac{a_2}{a_j}V_2 - \cdots - \frac{a_{j-1}}{a_j}V_{j-1}$. $\Rightarrow V_j$ is a linear combination of $[v_1, v_2, \ldots, v_{j-1}]$.

$\Rightarrow V_j \in \langle v_1, \ldots, v_{j-1} \rangle$

(b) We need to prove it two directions.

$\subseteq$). Since $[v_1, \ldots, \hat{V_j}, \ldots V_k] \subseteq \{v_1, \ldots, V_k\}$ then $\langle V_1, \ldots, \hat{V_j}, \ldots V_k \rangle = \langle V_1, \ldots, V_k \rangle$ in particular

5
(1.2)

$\supseteq$) Let's take any $v \in V$. such that $v = a_1V_1 + \cdots + a_kV_k$.

Since we know $V_j \in \langle V_1, \ldots, V_{j-1} \rangle \Rightarrow V_j = b_1V_1 + \cdots + b_{j-1}V_{j-1}$

Then $v = a_1V_1 + \cdots + a_{j-1}V_{j-1} + a_j(b_1V_1 + \cdots + b_{j-1}V_{j-1}) + \cdots + a_kV_k$

$v = (a_1 + a_jb_1)V_1 + (a_2 + a_jb_2)v_2 + \cdots + (a_{j-1} + b_{j-1})V_{j-1} + \cancel{a_jV_j} + a_{j+1}V_{j+1} + \cdots + a_kV_k$

Then $v \in \langle v_1, \dots, \hat{v_j}, \dots, v_k \rangle$

Thus $\langle v_1, \dots, \hat{v_j}, \dots, v_k \rangle = \langle v_1, \dots, v_k \rangle$

(c) By definition, every finite dimensional vector space has a spanning list. Then we consider two cases.

~~1. If~~ 1. The spanning list is linearly independent, then by definition of basis, this spanning list is a basis

2. The spanning list is ~~not a~~ linearly dependent,

let $v_1, \dots, v_k$ be a spanning list. ~~Use~~

Use promblem (1) (b). $\langle v_1, \dots, \hat{v_j}, \dots, v_k \rangle = \langle v_1, \dots, v_k \rangle$.

Then ~~the~~ if $v_1, \dots, \hat{v_j}, \dots, v_k$ is linearly independent, then we are done

if $v_1, \dots, \hat{v_j}, \dots, v_k$ is linearly dependent, then we can also do the process above.

Inductively, we can get a linearly independent list that is a basis!

If the list is ϕ, then the finite dimensional vector space is $\{0\}$. satisfied.

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(1.3)

PROBLEM 2 (20 POINTS). **The fundamental theorem of linear maps.**

In each case define a linear map satisfying the requirements. [No need to use matrices.] Then evaluate the map you defined in the given vector.

(a) $T : \mathbb{C}^4 \to \mathbb{C}^5$ such that

$$(1, 0, i, 0) \in \operatorname{Null} T; \quad T(0, 1 + i, 0, 0) = T(1 + i, 0, 0, 0) \neq 0; \quad (1, 0, i, 0, 1) \notin \operatorname{Range} T.$$

(b) Compute $T(1, 1, 1, 1)$.

(c) $R : \mathbb{R}_4[x] \to \mathbb{R}_3[x]$ such that

$$\dim(\operatorname{Null} R) = 2; \quad 1 + x + x^2 + x^3 \in \operatorname{Range} R; \quad 1 - x^4 \notin \operatorname{Null} R.$$

(d) Compute $R(x^4)$.

(a) Since $(1,0,i,0) \in \operatorname{Null} T$, ~~then $T(1,0,i,0) = (-ix+z, -iy, 0,0,0)$~~

~~which also satisfy $T(0,1+i,0,0) = T(1+i,0,0,0) = (1-i, 1-i, 0,0,0)$~~

And $T(x,y,z,w) = (a_{11}x + a_{12}y + a_{13}z + a_{14}w, \dots, a_{41}x + a_{42}y + a_{43}z + a_{44}w)$

then $x=1, z=i, y=w=0 \Rightarrow a_{11} = -ia_{13},\ a_{21} = -ia_{23},\ a_{31} = -ia_{33},\ a_{41} = -ia_{43}$

Then $T(x,y,z,w) = (-ia_{13}x + a_{12}y + a_{13}z + a_{14}w, \dots)$

$T(0,1+i,0,0) = T(1+i,0,0,0) \neq 0$. then $(a_{12} + ia_{12}, a_{32} + ia_{22}, \dots) = (-ia_{13} + a_{13}, \dots)$

$a_{12} + ia_{12} = -ia_{13} + a_{13}$ ~~$\Rightarrow a_{12} = a_{13},\ a_{12} = a_{13} \Rightarrow a_{12} = a_{13} = \dots$~~

~~then $T(x,y,z,w) = (a_{14}w, a_{24}w, a_{34}w, a_{44}w, a_{54}w)$~~

~~Since $(1,0,i,0,1) \notin$ Range T. We can define $T(x,y,z,w) = (w,0,0,0,0)$~~

Then let's $a_{12} = -i$, $a_{13} = 1$. We have $T(x,y,z,w) = (-ix - iy + z + a_{14}w, \dots)$

let $a_{14} = a_{24} = a_{34} \dots = a_{54} = 0$. (2.1)

Then $T(x,y,z,w) = (-ix - iy + z, ix - iy + z, \dots, ix - iy + z)$

(b) $T(1,1,1,1) = (-i - i + 1, \dots, -i - i + 1) = (-2i+1, -2i+1, -2i+1, -2i+1, -2i+1)$ (2.2)

(c). ~~Since $\dim(V) = \dim(\text{Null}) + \dim(\text{Range})$, the dim (range) $= 3$~~

~~$R(a + bx + cx^2 + dx^3 + ex^4) = a(1+x) + bx^2 + cx^3$~~ $+ b(x^2+x+1) + cx^3$

Then let basis of range be $\{1+x, x^2+x+1, x^3\}$ ~~$a=0, b=1, c=1$ satisfy $1+x+x^2+x^3 \in$ Range R.~~

$R(1-x^4) = 1+x \neq 0$. ✓ $b-c$ $(a)(x+1) + bx^2 + cx^3 = 0$, $\dim(\operatorname{Null} R) = 2$. ✓

(d) ~~$R(x^4) = 0$~~

let $R(a+bx+cx^2+dx^3+ex^4) = a(1+x)+(b-a)(x^2+x+1)+bx$

Null R:

$a(1+x)+(b-a)(x^2+x+1)+bx^3 = 0. \Rightarrow \begin{cases} a=0 \\ b=a \\ b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0. \end{cases}$

$\dim(\text{Null} R) = 2$ ✓

+5 (2.3)

why do you define this linear map? you must explain the process

let $a=0$, $b=1 \Rightarrow 1+x+x^2 \cancel{+x+1} + x^3 \in \text{range} R$ ✓.

$R(1-x^4) = \cancel{1+x \notin \text{Null} R}$ $1+x-x^2-x-1 = -x \notin \text{Null} R$

Thus $R(a+bx+cx^2+dx^3+ex^4) = a(1+x)+(b-a)(x^2+x+1)+bx^3$

is a valid linear map

(d) $R(x^4) = 0$ ✓ (2.4)

PROBLEM 3 (20 POINTS). **True / False.**

In each case chose **T** or **F** and prove your statement.

(a) The matrix $A = \begin{pmatrix} 5 & 6 & 6 & 8 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{pmatrix}$ is invertible and the inverse of A is given by:

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -68 & -36 & 48 & 64 \\ 68 & 35 & -47 & -64 \\ -16 & -9 & 11 & 16 \\ 4 & 3 & -3 & 4 \end{pmatrix}.$$

(b) The span of three vectors can never contain the span of four different non-zero vectors.

(c) If two linear maps $T_1, T_2 \colon \mathbb{R}^n \to \mathbb{R}^m$ have the same null space and the same range, then they are the same linear map.

(d) Let V and W be finite dimensional complex vector spaces. Then there exists a bijective linear map between the complex vector spaces $L(V,W)$ and $L(W,V)$.

(a). $\cancel{A = \left(\begin{array}{cccc|cccc} 5 & 6 & 6 & 8 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 8 & 0 & 1 & 0 & 0 \\ 6 & 6 & 2 & 8 & 0 & 0 & 1 & 0 \\ 2 & 3 & 6 & 7 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{cccc|cccc} 5 & 6 & 6 & 8 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 8 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & -16 & 0 & -3 & 1 & 0 \\ 2 & 3 & 6 & 7 & 0 & 0 & 0 & 1 \end{array}\right)}$

let $A \cdot A^{-1} = \begin{pmatrix} 5 & 6 & 6 & 8 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} -17 & -9 & 12 & 16 \\ 17 & \frac{35}{4} & -\frac{47}{4} & -16 \\ -4 & -\frac{9}{4} & \frac{11}{4} & 4 \\ 1 & \frac{3}{4} & -\frac{3}{4} & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 16 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$ Since $a_{14} = 16 \neq 0$, thus $A \times A^{-1}$ is not identity matrix, thus False

(3.1)

(b) False. counterexample. $\langle (1,0,0),(0,1,0),(0,0,1),(1,1,1) \rangle \subseteq \langle (0,0,1),(0,1,0),(0,0,1) \rangle$. Since they all spans $\mathbb{R}^3$. ~~in particular~~

(3.2)

(c) ~~Since dim(V) = dim(null) + dim(range), then, T_1, T_2~~

False, counter example. $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$. $T_1(x,y) = (x+y, x-y)$

$T_2(x,y) = (x-y, x+y)$

$\text{Null}(T_1) = \{0\} = \text{Null}(T_2)$

$\text{range}(T_1) = \langle (1,1),(1,-1) \rangle = \mathbb{R}^2$

$\text{range}(T_2) = \langle (1,1),(-1,1) \rangle = \mathbb{R}^2$

Then $\text{range}(T_1) = \text{range}(T_2)$ But $T_1 \neq T_2$

5 (3.3)

(d) ~~True~~ ~~False~~

~~Let $L(V,W) = L(0,\mathbb{R})$~~
~~$L(W,V) = L(\mathbb{R},0)$~~
~~$T: V \to W$, $T(0) = X$, $X \in \mathbb{R}$.~~
~~$\mathbb{R}: W \to V$ $T(X) = 0$.~~

True NTP. $T: L(V,W) \to L(W,V)$ is bijective.

$\forall M \in L(W,V)$, $\exists!$ ~~$L(V,W)$~~ $R \in L(V,W)$ satisfied.

$\forall M: W \to V$, we know $M^{-1}: V \to W$ is a invertible of M.

which satisfy M is injective and surjective. $\Rightarrow$ bijective.

Then for any $M \in L(W,V)$, we all can find. M^{-1}

which M is ~~bijective~~ surjective. ~~Since for $V \to W$. by theorem there exists unique linear map from~~

Then. $M^{-1} = R$.

~~Then $T: L(V,W) \to L(W,V)$ is bijective~~.

Since by theorem, 6 from $M \to M^{-1}$, there exists a unique linear map.

0
(3.4)

PROBLEM 4 (20 POINTS). **Systems of linear equations.**

Consider the following systems of linear equations over $\mathbb{Z}_5$.

$$\begin{cases} 2x+y+z=4 \\ 4x+y+z=1 \\ \quad\quad y+z=a \end{cases} \qquad \begin{cases} 2x+y+z=0 \\ 4x+y+z=0 \\ \quad\quad y+z=0 \end{cases}$$

(a) Find a basis of the solutions subspace of the homogeneous one.

(b) Determine for which values of a, the non-homogeneous system has a solution. In this case, describe the affine space of all solutions.

Write in matrix

(a) $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Then $y+z=0 \Rightarrow y=-z$. Thus $(x,y,z) = \{(0,-z,z) \mid z \in \mathbb{Z}_5\}$

Thus the basis is $\cancel{\{(1,0,0),(0,4,1)\}}$ $\{(0,-1,1)\}$ **+9** **(4.1)** you must give the answer in Z5

(b) ~~Since we have the non-reduced echelon form~~

~~Then~~ $\cancel{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$ ~~And~~ Do non-reduced echelon operation again

$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 4 & 1 & 1 & 1 \\ 0 & 1 & 1 & a \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & a \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 0 & 0 & 4-a \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & 2-a \end{array}\right)$

then by theorem, $2-a=0 \Rightarrow a=2 \pmod 5$

Thus $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ~~⇒ (, since we are in $\mathbb{Z}_5$. then~~ ,

~~$2\times 3 = 6 \equiv 1 \pmod 5$~~ $\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x=1 \\ y+z=2 \end{cases} \quad X_0=(1,2,0)$

Then the affine space is $\cancel{X_0+W=\{(1,2,0)+\langle(1,0,0),(0,4,1)\rangle\}}$

(4.2)

$X_0+W=\{(1,2,0)+\langle(0,-1,1)\rangle\}$

PROBLEM 5 (20 POINTS). **On subspaces.**

Let $V = M_{n\times n}(\mathbb{R})$ and let A be a fixed matrix in V. Let $W = \{B \in V : AB + B^tA = 0\}$

(a) Prove that W is a subspace of V.

(b) Let $n = 2$ and let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Determine W explicitely.

(c) Let W_1 and W_2 be the following subspaces of V.

$$W_1 = \left\{ \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}; \qquad W_2 = \left\{ \begin{pmatrix} a & -b \\ a & -b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

One of these is a direct complement of the subspace $\langle A \rangle + W$. Which one?

(a) we need check 3 things.

① $0 \in W$

~~since~~ If $B=0$, then $AB+B^tA = A\cdot 0 + 0\cdot A = 0+0 = 0$. ✓

② $v, w \in W$, $v+w \in W$

let take $B_1 : AB_1 + B_1^tA = 0$, $B_2 : AB_2 + B_2^t \cdot A = 0$.

6
(5.1)

Then we plus this two equation $\Rightarrow AB_1 + B_1^tA + AB_2 + B_2^tA = 0$

$\Rightarrow A(B_1+B_2) + (B_1^t + B_2^t)A = 0 \Rightarrow B_1+B_2 \in W$ ✓ ✓.

③ $\lambda \in \mathbb{R}$, $v \in W$, $\lambda v \in W$.

let take $B : AB + B^tA = 0$. Multiply $\lambda \in \mathbb{R}$ $\Rightarrow \lambda(AB+B^tA) = \lambda \cdot 0 = 0$

$\Rightarrow \lambda AB + \lambda B^tA = 0$ ~~$\Rightarrow A(\lambda B)$~~ $\Rightarrow (\lambda I)AB + (\lambda I)B^tA = 0 \Rightarrow A(\lambda I)B + (\lambda I)B^tA = 0$

$\Rightarrow A(\lambda B) + (\lambda B^t)A = 0 \Rightarrow \lambda B \in W$ ✓ ✓

Thus W is a subspace of V

(b) let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ since $AB + B^tA = 0$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$

let's compute left side: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} c & d \\ a & b \end{pmatrix} + \begin{pmatrix} c & a \\ d & b \end{pmatrix} = \begin{pmatrix} 2c & a+d \\ a+d & 2b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

7
(5.2)

✓

Then $\begin{cases} c = 0 \\ a+d = 0 \\ b = 0. \end{cases} \Rightarrow \begin{cases} b = c = 0 \\ a = -d. \end{cases}$ Then $B = \begin{pmatrix} -d & 0 \\ 0 & d \end{pmatrix}$. Thus $W = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$

~~we~~ Since we need a direct complement to $\langle A\rangle + W$, let's
this V, and V is either W_1 or W_2

Then $V \oplus (\langle A\rangle + W) = M_{2\times 2}(\mathbb{R})$

~~Since~~ We need ~~compute~~ $\langle A\rangle + W = \langle \begin{pmatrix}0&1\\1&0\end{pmatrix}\rangle + \langle \begin{pmatrix}-1&0\\0&1\end{pmatrix}\rangle$

$\cancel{= \begin{pmatrix}0&\lambda\\ \lambda&0\end{pmatrix} + \begin{pmatrix}-\mu&0\\0&\mu\end{pmatrix} = \begin{pmatrix}-\mu&\lambda\\ \lambda&\mu\end{pmatrix} \quad (\lambda,\mu\in\mathbb{R})}$

~~So~~ let's try $W_1 = \left\{\begin{pmatrix}a&-b\\-b&a\end{pmatrix} : a,b\in\mathbb{R}\right\} \Rightarrow W_1 = \langle \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\rangle$

7
(5.3)

But ~~$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is same~~ $\begin{pmatrix}0&1\\1&0\end{pmatrix} \in \langle A\rangle + W$. ~~thus~~

And ~~$M_{2\times2}(\mathbb{R})$ need~~ $\dim(M_{2\times2}(\mathbb{R})) = 4$ Then W_1 is not a direct complement.

Claim: W_2 is a direct complement ✓

Let's try $W_2 = \left\{\begin{pmatrix}a&-b\\a&-b\end{pmatrix} : a,b\in\mathbb{R}\right\} = \langle \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}\rangle$.

Then $W_2 + \langle A\rangle + W = \langle \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&1\end{pmatrix}\rangle$.

We need to prove it is a basis to $M_{2\times2}(\mathbb{R})$

① generates $M_{2\times2}(\mathbb{R})$

let $A \in M_{2\times2}(\mathbb{R})$, $A = \begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}x&0\\x&0\end{pmatrix} + \begin{pmatrix}0&y\\0&y\end{pmatrix} + \begin{pmatrix}0&z\\z&0\end{pmatrix} + \begin{pmatrix}-w&0\\0&w\end{pmatrix}$

$= \begin{pmatrix}x-w & y+z\\ x+z & y+w\end{pmatrix}$

$\Rightarrow \begin{cases} a = x-w \\ b = y+z \\ c = x+z \\ d = y+w \end{cases} \Rightarrow \cancel{\begin{cases} x = -w-a \\ y = z-b \\ z = x-c \\ w = d-y \end{cases}} \Rightarrow \cancel{\{ x = y-d-a} \Rightarrow \begin{cases} x = w+a \\ y = b-z \\ z = c-x \\ w = d-y \end{cases} \Rightarrow \begin{cases} x = a+d-y \\ y = b-c+x \\ z = c-x \\ w = d-y \end{cases}$

$\Rightarrow \cancel{\begin{cases} a+c = \ldots \\ a+b = \end{cases}}$

then $x = a+d-(b-c+x) = a+d-b+c-x \Rightarrow 2x = a+d-b+c \Rightarrow x = \frac{a+d-b+c}{2}$

$y = \frac{2b-2c+a+d-b+c}{2} = \frac{b-c+a+d}{2} \Rightarrow z = \frac{2c-a-d+b-c}{2} = \frac{c+b-a-d}{2} \Rightarrow w = \frac{2d-b+c-a-d}{2}$

$= \frac{d-b+c-a}{2}$

thus it spans $M_{2\times2}(\mathbb{R})$

② linearly independent.

$0 = \begin{pmatrix}x&0\\x&0\end{pmatrix} + \begin{pmatrix}0&y\\0&y\end{pmatrix} + \begin{pmatrix}0&z\\z&0\end{pmatrix} + \begin{pmatrix}-w&0\\0&w\end{pmatrix} = \begin{pmatrix}x-w & y+z\\ x+z & y+w\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix} \Rightarrow \begin{cases} x = w \\ y = -z \\ y = -w \\ x = -z \end{cases} = \begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ w = 0 \end{cases}$ ✓.