Practice Exercises

August 23, 2024

1. By constructing an injection from $\mathbb{N}$ so X show that each of the following sets is infinite:

   (i) $\mathbb{Z}$.

   (ii) $\{x \in \mathbb{Z} : x < 0\}$.

   (iii) $\{n \in \mathbb{N} : n \geq 10^6\}$.

2. Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

   is a bijection.

3. Let X be a non-empty subset of $\mathbb{Z}$ which has no least member. Show that we can choose a sequence $x_1, x_2, \ldots, x_n, \ldots$ of distinct members of X such that $x_n < x_{n-1}$ for all $n \in \mathbb{N}$. Deduce that X is infinite.

4. In the lectures, it's been seen that it may happen that $Y \subsetneq X$ while $|X| = |Y|$. Show that this is not possible if X is finite.

5. Prove that given any integer $n \geq 1$, there exists an odd integer m and integer $k \geq 0$ such that $n = 2^k m$.

6. Find $d = \gcd(234, 63)$ and write it as a linear combination of 234 and 63.

7. Let A_1 and A_2 be two disjoint finite sets. Then $|A_1 \cup A_2| = |A_1| + |A_2|$.

8. Let $n \in \mathbb{N}$. Let $A_1, A_2, \ldots A_n$ be n pairwise disjoint finite sets. Then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.

9. Let A_1 and A_2 be two finite sets. Then $|A_1 \times A_2| = |A_1| \cdot |A_2|$.

10. If X and Y are infinite countable and disjoint sets, then $X \cup Y$ is countable.

11. Let X be an infinite countable set and let Y be a finite set with empty intersection. Then $X \cup Y$ is countable.

12. Let X be a countable set and let Y be a countable set. Then $X \cup Y$ is countable.

13. Let $n \in \mathbb{N}$. If X_1, $X_2, \ldots X_n$ are countable sets. Then $\bigcup_{i=1}^{n} X_i$ is countable.

14. On the cartesian product of finite sets:

    (a) Let A and B be two finite sets with $|A| = m$ and $|B| = n$. Prove that $A \times B$ is finite and moreover

$$|A \times B| = m \cdot n = |A| \cdot |B|.$$

    (b) Let $A_1, A_2, \ldots, A_n$ be a finite collection of finite sets. Prove by induction (use item (a)) that the cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is finite and moreover

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|.$$

[Hint: You may use the result on cardinality of the finite union of finite sets.]

15. Let $(A_i)_{i \in \mathbb{N}}$ be any sequence of nonempty finite sets. Is their cartesian product, $A_1 \times A_2 \times A_3 \times \cdots$, countable?[1] Justify your answer.

16. (The Hanoi tower problem) There are three poles, and on one of them are n discs of different sizes, stacked in order of size with the largest at the bottom. The task is to move the stack to another pole, subject to the following rules:

    (a) Only one disc may be moved at a time.

    (b) A disc may never be placed on top of a smaller one.

    How many moves are needed to transfer the stack to another pole? Guess the answer and prove it by induction.

17. Prove that $11^n - 6$ is divisible by 5 for all positive integer n.

18. Prove that

    $$1/2 + 1/4 + 1/8 + \cdots + 1/2^n = 1 - 1/2^n$$

    for any positive integer n.

19. Write the first five terms of each of the following sequences:

    (a) $a_n = 2n$

    (b) $a_n = (-1)^{n+1} n^2$

    (c) $a_n = \frac{1}{n} \sin(n\frac{\pi}{2})$

    (d) $a_n = \frac{\sqrt{n}}{n+1}$

    (e) $a_n = \frac{2^{n-1}}{(2n-1)^3}$

    (f) $a_n = 2a_{n-1}$, with $a_0 = 3$

    (g) $a_n = 3a_{n-2}$, with $a_0 = 1$ and $a_1 = 3$

    (h) $a_n = a_{n-1}^2$, with $a_0 = \pi$

20. Prove that for any $n \in \mathbb{N}_{>0}$, $6^n - 1$ is divisible by 5.

21. Prove that $\forall n \in \mathbb{N} : \sum_{i=1}^{n} (2i - 1) = n^2$ by using the Gauss's sum and by the inductive method.

22. Prove that $\forall n \in \mathbb{N}$ we have

    (a) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

    (b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

23. Prove that $\forall n \in \mathbb{N}$ we have

    (a) $\displaystyle\sum_{i=1}^{n} (-1)^{i+1} i^2 = (-1)^{n+1} \frac{n(n+1)}{2}$

    (b) $\displaystyle\sum_{i=1}^{n} (2i+1)\, 3^{i-1} = n3^n$

    (c) $\displaystyle\prod_{i=1}^{n} \left(1 + a^{2^{i-1}}\right) = \frac{1 - a^{2^n}}{1 - a}$, $a \in \mathbb{R} \setminus 1$

[1] Recall that $A_1 \times A_2 \times A_3 \times \cdots$ is the set of sequences $(a_i)_{i \in \mathbb{N}}$ with $a_i \in A_i$ for all $i \in \mathbb{N}$

24. (a) Let $(a_n)_{n\in\mathbb{N}}$ be sequence of real numbers. Prove that $\sum_{i=1}^{n}(a_{i+1}-a_i)=a_{n+1}-a_1$

(b) Compute $\displaystyle\sum_{i=1}^{n}\frac{1}{i\,(i+1)}$

(c) Compute $\displaystyle\sum_{i=1}^{n}\frac{1}{(2i-1)\,(2i+1)}$

25. Prove that the following inequalities are valid $\forall n\in\mathbb{N}$

(a) $3^n+5^n\geq 2^{n+2}$

(b) $3^n\geq n^3$

(c) $\displaystyle\sum_{i=1}^{n}\frac{n+i}{i+1}\leq 1+n\,(n-1)$

(d) $\displaystyle\sum_{i=n}^{2n}\frac{i}{2^i}\leq n$

(e) $\displaystyle\sum_{i=1}^{2^n}\frac{1}{2i-1}>\frac{n+3}{4}$

(f) $\displaystyle\sum_{i=1}^{n}\frac{1}{i!}\leq 2-\frac{1}{2^{n-1}}$

(g) $\displaystyle\prod_{i=1}^{n}\frac{4i-1}{n+i}\geq 1$

26. Prove that

(a) $n!\geq 3^{n-1},\quad \forall n\geq 5$

(b) $3^n-2^n>n^3,\quad \forall n\geq 4$

(c) $\displaystyle\sum_{i=1}^{n}\frac{3^i}{i!}<6n-5,\quad \forall n\geq 3$

27. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers defined recursively as

$$a_1=2,\qquad a_{n+1}=2na_n+2^{n+1}n!,\quad \forall n\in\mathbb{N}$$

Prove that $a_n=2^n n!$.

28. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers defined recursively as

$$a_1=0\qquad a_{n+1}=a_n+n\,(3n+1)\,,\quad \forall n\in\mathbb{N}$$

Prove that $a_n=n^2\,(n-1)$.

29. Prove that any natural number n can be written as a sum of distinct powers of 2 (including $2^0=1$).

30. Let n_k be the number of ways to select a pair of elements out of a set of k elements. Give a recursive formula for n_k.

31. Use induction to show that

$$a+ar+ar^2+\cdots+ar^n=a\left(\frac{r^{n+1}}{r-1}\right)$$

for $r\neq 1$ and for all $n\geq 0$.

32. Use strong induction to show that if the sequence a_n is defined recursively by

$$a_1=3,\ a_2=5 \text{ and } a_n=3a_{n-1}-2a_{n-2} \text{ for } n\geq 3$$

then $a_n=2^n+1$ for every $n\in\mathbb{N}$.

33. (Exercise 3, from Biggs 1.1) Prove that if any two integers a and b are given, then there is an integer c such that $(a+b)c = a^2 - b^2$.

34. (Exercise 2, from Biggs 1.2) Show that $0 \leq x^2$ for any x in $\mathbb{Z}$, and deduce that $0 \leq 1$.

35. (Exercise 3, from Biggs 1.2) Deduce from the previous exercise that $n \leq n+1$ for all $n \in \mathbb{Z}$.

36. (Exercise 2, from Biggs 1.3) Give a recursive definition of the "n-th power" 2^n for all $n \geq 1$.

37. Show that if $n \geq 2$ and n is not prime then there is a prime p such that $p|n$ and $p^2 \leq n$.

38. Prove that 467 is prime.

39. Divide ± 25 by ± 3 using the Euclidean algorithm.

40. Let $m \in \mathbb{N}$. Show that $m \cdot n \geq m$ for all $n \in \mathbb{N}$.

    [Hint: use induction on n.]

    As a consequence, given $m, n \in \mathbb{N}$, if $m \cdot n = 1$ then $m = n = 1$.

41. Show that $4^{2n} - 1$ is divisible by 15 for all $n \in \mathbb{N}$.

42. Show that if a and b are integers such that $ab = 1$ then $a = b = 1$ or $a = b = -1$.

    [Hint: either both a and b are positive or both are negative.]

    Deduce that if x and y are integers such that $x|y$ and $y|x$ then $x = \pm y$. In particular, if x divides 1 then $x = \pm 1$.

43. We know that the order relation "$\leq$" on $\mathbb{Z}$ satisfies certain axioms, which are the following:

    (i) $a \leq a$ for all $a \in \mathbb{Z}$. *(Reflexivity)*

    (ii) If $a \leq b$ and $b \leq a$, then $a = b$ for any $a, b \in \mathbb{Z}$. *(Antisymmetry)*

    (iii) If $a \leq b$ and $b \leq c$, then $a \leq c$, for any $a, b, c \in \mathbb{Z}$. *(Transitivity)*

    In general, given a set X and a relation $\sim$ on X, we say that $\sim$ is an *order relation* if it satisfies the following axioms:

    (i) $a \sim a$ for all $a \in X$. *(Reflexivity)*

    (ii) If $a \sim b$ and $b \sim a$, then $a = b$ for any $a, b \in X$. *(Antisymmetry)*

    (iii) If $a \sim b$ and $b \sim c$, then $a \sim c$, for any $a, b, c \in X$. *(Transitivity)*

    Prove that the divisibility relation "$|$" on $\mathbb{N}$ is an order relation.

44. If $c|a$ and $c|b$, then $c|xa + yb$ for any $x, y \in \mathbb{Z}$. Prove this fact.

45. Let $a, b \in \mathbb{Z}$. Given $n \in \mathbb{N}$, then $n|(b-a)$ if and only if the remainder of a divided by n is the same as the remainder of b divided by n. That is, if the difference between a and b is a multiple a and b share the same remainder when divided by n, and vice versa.

46. Let a and b be positive integers and let $d = \gcd(a, b)$. Prove that there are integers x and y for which $ax + by = c$ if and only if $d|c$.

47. Find the greatest common divisor of $a = 1320$ and $b = 714$ and express the result in the form $ax + by$ for some $x, y \in \mathbb{Z}$.

48. Show that in any set of 12 integers there are two whose difference is divisible by 11.

49. Show that 725 and 441 are coprime and find integers x and y such that $725x + 441y = 1$.

50. Let $d = \gcd(a, b)$. Which are the possible values for $\gcd(a + b, 2b - a)$? Give examples in which $\gcd(a + b, 2b - a)$ equals each of these possibilities.

51. Prove the following properties of the greatest common divisor:

    (a) If $m \in \mathbb{N}$ then $\gcd(ma, mb) = m \cdot \gcd(a, b)$.

    (b) If $\gcd(a, x) = d$ and $\gcd(b, x) = 1$ then $\gcd(ab, x) = d$.

52. If p and p' are distinct primes, prove that p does not divide p'.

53. Let $x, y \in \mathbb{Z}$. Prove that if $x^2 + y^2$ is divisible by 3 then x and y are both divisible by 3.

54. Prove that there are no integers $x, y, z, t \in \mathbb{Z}$ for which $x^2 + y^2 - 3z^2 - 3t^2 = 0$ unless all of them are 0.