Practice Exercises

August 14, 2024

1. Let p stand for the proposition "The weather is cloudy" and q for "It's raining". Express the following as natural English sentences:

   (a) $\neg p$

   (b) $p \vee q$

   (c) $p \wedge q$

   (d) $p \Rightarrow q$

   (e) $\neg p \Rightarrow \neg q$

   (f) $\neg p \vee (p \wedge q)$

2. Let p and q be two propositions. For each of the following propositions, construct a truth table and state whether the proposition is valid or satisfiable.

   (a) $p \wedge q$

   (b) $p \wedge \neg p$

   (c) $p \vee \neg p$

   (d) $(p \vee \neg q) \Rightarrow q$

   (e) $(p \vee q) \Rightarrow (p \wedge q)$

   (f) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

3. Determine whether the following two statements are logically equivalent or not. Justify your answer.

   (a) $\neg(p \Rightarrow q)$ and $p \wedge \neg q$

   (b) $p \Rightarrow (q \vee r)$ and $(p \Rightarrow q) \vee (p \Rightarrow r)$

4. Define appropriate propositional functions and translate the following statements into symbolic language:

   (a) All students are smart.

   (b) There exists a smart student.

   (c) Every student is friends with another student.

   (d) Bob is a student.

   (e) There exists a student that is friends with every other student.

   (f) Bob takes either Analysis or Geometry (but not both).

5. Translate the following into a natural English sentence.

   (a) $\exists z : p(z,x) \wedge s(z,y) \wedge w(y)$, where $p(z,x)$ means z is a parent of x, $s(z,y)$ means z and y are siblings, $w(y)$ means y is a woman, and x, y, and z range over people.

6. Determine the truth value of the following proposition:

   (a) $2^3 = 8$, if and only if, 49 is a perfect square.

   (b) $\pi = \dfrac{22}{7}$, if and only if, $\sqrt{2}$ is a rational number.

7. What can be said about the truth value of q when
   (a) p is false and $p \Rightarrow q$ is true?
   (b) p is true and $p \Leftrightarrow q$ is false?
8. Prove that for all $x \in \mathbb{R}$, if x^2 is irrational then x is irrational. (Using contrapositive)
9. Let p and q the statements
$$p \equiv n^2 \text{ is even}, \qquad q \equiv n \text{ is even}$$
   prove that $p \Rightarrow q$. (Using contradiction)
10. Prove that
$$(\exists x)(p(x) \wedge q(x)) \Rightarrow (\exists x)(p(x)) \wedge (\exists x)(q(x))$$
   and refute the other implication.
11. Prove that the existentially quantified statement
$$(\exists x)\left(\frac{1}{x^2+1} > 1\right)$$
   is false where the domain of the propositional function $p(x) \equiv \dfrac{1}{x^2+1}$ is $\mathbb{R}$.
12. Which of the following are denials of $(\exists! x)p(x)$? (there exists more than one solution)
   (a) $(\forall x)p(x) \vee (\forall x)\neg p(x)$.
   (b) $(\forall x)\neg p(x) \vee (\exists y)(\exists z)\big((y \neq z) \wedge p(y) \wedge p(z)\big)$
   (c) $(\forall x)\Big[p(x) \Rightarrow (\exists y)\big(p(y) \wedge (y \neq x)\big)\Big]$.
   (d) $\neg(\forall x)(\forall y)\Big[\big(p(x) \wedge p(y)\big) \Rightarrow (y = x)\Big]$
13. Prove that $(\exists! x)p(x)$ is equivalent to $(\exists x)\big[p(x) \wedge (\forall y)(p(y) \Rightarrow (y = x))\big]$. (Hint: Recall last problem)
14. Let x, y and z be real numbers. Prove that
   (a) If $x^3 + 2x^2 < 0$, then $2x + 5 < 11$.
   (b) If an isosceles triangle has sides of lengh x, y and z, where $x = y$ and $z = \sqrt{2}x$, then it is a right triangle.
15. Decide the merit of the following claims and the validity of the proof and then assign a grade of
   (a) **A** (correct) if the claim and proof are correct, even if the proof is not the simplest or the proof you would have given.
   (b) **B** (partially correct) if the claim is correct *and* the proof is largely correct. The proof may contain one or two incorrect statements or justifications, but the errors are easily corectable.
   (c) **C** (failure) if the claim is incorrect, or the main idea of the proof is incorrect, or there are too many errors.

   You must justify assigments of grades other than A and if the proof is incorrect, explain what is incorrect and why.

(a) Suppose a is an integer.
Claim: If a is odd, then $a^2 + 1$ is even.
"Proof": Let a. Then, by squaring an odd we get an odd. An odd plus odd is even. So, $a^2 + 1$ is even.

(b) Suppose a, b and c are integers.
Claim: If a divides b and a divides c, then a divides $b + c$.
"Proof": Suppose a divides b and a divides c. Then, for some integer q, $b = aq$, and for some integer q, $c = aq$. Then, $b + c = aq + aq = 2aq = a(2q)$, so a divides $b + c$.

(c) Suppose m is an integer.
Claim: If m^2 is odd, then m is odd.
"Proof": Assume m is odd. Then $m = 2k + 1$ for some integer k. Therefore, $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. Therefore, if m^2 is odd, then m is odd.

16. Prove that for every real number x, if $x^3 + x > 0$, then $x > 0$.

17. Suppose that a circle has center $(2, 4)$. Prove that if $(0, 3)$ is not inside the circle, then $(3, 1)$ is not inside the circle.

18. Three real numbers x, y and z are chosen between 0 and 1 with $0 < x < y < z < 1$. Prove that at least two of the numbers x, y and z are within $\frac{1}{2}$ unit from one another.

19. Prove that if every even natural number greater than 2 is the sum of two primes, [1] then every odd natural number greater than 5 is the sum of three primes.

20. True or False: For all positive real numbers x, $2^x > x + 1$.

21. True or False: For every positive real number x, there is a positive real number y less than x with the property that for all positive real numbers z, $yz \geq z$.

22. Prove that if p is a prime number and $p \neq 3$, then 3 divides $p^2 + 2$.
(Hint: When p is divided by 3, the remainder is either 0,1 or 2. That is, for some integer k, $p = 3k$ or $p = 3k + 1$ or $p = 3k + 2$).

23. Prove that there is no largest natural number.

24. Prove that for every natural number x there is an integer k such that $3.3x + k < 50$

25. Let A, B and C be three sets as shown in the following Venn diagram.

    For each of the following sets, draw a Venn diagram and shade the area representing the given set.

    (a) $A \cup B \cup C$

    (b) $A \cap B \cap C$

    (c) $A \cup (B \cap C)$

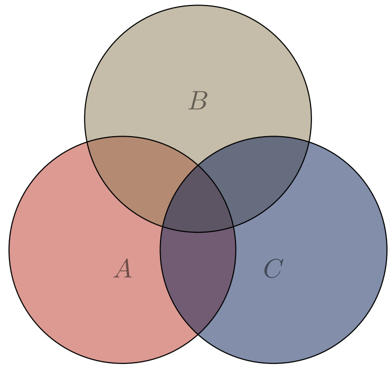
    (d) $A \setminus (B \cap C)$

    (e) $A \cup (B \cup C)^c$

    (f) $A^c \cap (B \cap C)$

26. Let A be a set defined as $A = \{1, 2, \{3\}, \{1, 2\}\}$. Determine which of the following asserts are correct.

[1] No one knows whether every even number greater than 2 is the sum of two prime numbers. This is the famous Goldbach Conjecture, proposed by the Prussian mathematician Christian Goldbach in 1742. You should search the Web to learn about the million dollar prize (never claimed) for proving Goldbach's Conjecture. Fortunately, you don't have to prove it in this exercise.



(a) $3 \in A$

(b) $\{3\} \in A$

(c) $\{3\} \subseteq A$

(d) $\{\{3\}\} \subseteq A$

(e) $\{1, 2\} \in A$

(f) $\{1, 2\} \subseteq A$

(g) $\{\{1, 2\}\} \subseteq A$

(h) $\{\{1, 2\}, 3\} \subseteq A$

(i) $\emptyset \in A$

(j) $\emptyset \subseteq A$

(k) $A \in A$

(l) $A \subseteq A$

27. Let $A, B, C \subseteq X$ be sets. Prove the following properties.

(a) $A \cup B = B \cup A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

(c) $A \cup \emptyset = A$

(d) $A \cup A = A$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) $A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$

(g) $A \subseteq A \cup B$

(h) $A \cap B = B \cap A$

(i) $(A \cap B) \cap C = A \cap (B \cap C)$

(j) $A \cap \emptyset = \emptyset$

(k) $A \cap A = A$

(l) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(m) $A \cup A^c = X$

(n) $A \subseteq B \Leftrightarrow B^c \subseteq A^c$

28. Write three possible partitions of the set $A = \{1, 2, 3, 5, 7, 9, 8\}$

29. The function $f : X \to Y$ is said to have a left inverse if there is a function $l : Y \to X$ such that $l \circ f = \mathrm{id}_X$. Show that

(a) if f has a left inverse then it is an injection;

(b) if f is an injection then it has a left inverse.

30. Let $s, t : \mathbb{Z} \to \mathbb{Z}$ be functions defined by $s(x) = x + 1$ and $t(x) = 2x$. Prove that $s \circ t \neq t \circ s$.

31. Let $X = \{1, 2, 3, 4, 5\}$ and let $f : X \to X$ be the function defined by $f(1) = 2$, $f(2) = 2$, $f(3) = 4$, $f(4) = 4$ and $f(5) = 4$. Show that $f \circ f = f$. Find a function $g \neq f$ such that $g \circ f = f$ and $f \circ g = f$.

32. If f and g are bijections and $g \circ f$ is defined, then the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.

33. The function $f : X \to Y$ is said to have a left inverse if there is a function $l : Y \to X$ such that $l \circ f = \mathrm{id}_X$. Show that

   (a) if f has a left inverse then it is an injection;

   (b) if f is an injection then it has a left inverse.

34. Formulate and prove results about a right inverse of $f : X \to Y$ which correspond to those in the previous exercise.

35. Let us refer to the following subsets of $\mathbb{Z}$ as *blocks*:

$$B_1 = \{1,2,4\},\ B_2 = \{2,3,5\},\ B_3 = \{3,4,6\},\ B_4 = \{4,5,7\},$$
$$B_5 = \{1,5,6\},\ B_6 = \{2,6,7\},\ B_7 = \{1,3,7\}.$$

   Define

$$f(x,y) = \begin{cases} z & \text{if } x \neq y \text{ and } \{x,y,z\} \text{ is a block,} \\ x & \text{if } x = y. \end{cases}$$

   Is f a function? Is it an injection?

36. Suppose that $f : A \to B$ and $g : B \to C$ are functions. Prove that if $g \circ f$ is injective then f is injective, and if $g \circ f$ is surjective then g is surjective.

37. Let $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = x^2$. Show that f is injective.

38. Let $f : (-2, \infty) \to (-\infty, 4)$ the injective function given by $f(x) = \dfrac{4x}{x+2}$. Find its inverse function.

39. Show that the relation R defined on the positive integers $\mathbb{Z}^+$ by xRy if $x \mid y$, is a partial order.

40. If R is a relation on X, the **inverse** of R is the relation

$$R^{-1} = \{(y,x) : xRy\}.$$

   Prove that R is symmetric, if and only if, $R = R^{-1}$.